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HIGHER DIMENSIONAL GENERALIZATION OF TOLMAN'S SCHWARZSCHILD INTERIOR SOLUTION IN GENERAL RELATIVITY

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ABSTRACT

Here this paper is dedicated to find and generalize the famous Richard C. Tolman's Schwarzschild interior solution in higher dimensions. This solution is very important and useful in understanding and discussing the internal construction of stars. Max Wyman's generalization has been also discussed to some extent. Energy density ρ and pressure p have been found. These solutions can be matched at the boundary with the exterior solution (Myers and Perry, 1986).

Key words: dimension, stars, energy, density, Pressure, universe, field equation.

1. INTRODUCTION

Einstein's field equations which connect the distribution of matter and energy with the gravitational field (described by the potential g_{ij}) are

$$-8\pi T_{ij} = R_{ij} - \frac{1}{2} R g_{ij} + \wedge g_{ij}$$

where T_{ij} is the material energy momentum tensor, R_{ij}. The Ricci tensor, R, the curvature invariant (or scalar curvature tensor) and $\wedge$, the cosmological constant. The field equations being highly non-linear, the exact solutions are obtained only in a few special cases. Professor Richard C. Tolman [27] considered the problem which corresponds to an equilibrium distribution of perfect fluid and obtained solutions under some mathematical restrictions. Here we discuss and obtain higher dimensional generalization of Tolman's (1939) solution which are significant in the study of the internal structure of stars.

If we set $\wedge = 0$ in agreement with known fact that the cosmological constant is too small to produce appreciable effective within a moderate spatial range. The above Einstein's field equations (with $\wedge=0$) may be written as

$$-8\pi T_{ij} = R_{ij} - \frac{1}{2} R g_{ij}$$

Where velocity of light and gravitational constant is taken to be unity in the usual units.

Infect physicist shown their keen interest in the exact physical situation at very early stages of the formation of our universe. World geometry

by Kaluza-Klein is that the universe started out in a higher-dimensional phase with some dimensions eventually collapsing and stabilizing at a size close to the Planck length while three others continued to expand and are still doing so (Sahdev [21]). Here this paper is dedicated to find and generalize the famous Richard C. Tolman's Schwarzschild interior solution in higher dimensions. This solution is very important and useful in understanding and discussing the internal construction of stars. Max Wyman's generalization has been also discussed to some extent. Energy density ρ and pressure p have been found. These solutions can be matched at the boundary with the exterior solution (Myers and Perry, 1986).

2. THE FIELD EQUATIONS

We take the static spherically symmetric metric for D = n+3-dimensional space-time in the form given by (Myers and Perry [20].

(2.1)

where A = A(r) and B = B (r). Also

etc.

The Einstein field equations for D = (n + 3) dimensional space. time are

(2.2)

The field equations (2.2) for the metric (2.1) yield

(2.3)

$$e^{-A}\left(\frac{B'}{r}+\frac{n}{r^2}\right)-\frac{n}{r^2}=\frac{16\pi}{(n+1)}p$$

(2.4)

(2.5)

(2.6)

We multiply equation (2.4) by 2/r and then arranging the terms, we obtain

(2.7)

3. SOLUTIONS OF THE FIELD EQUATION

The general solution of equation (2.7) as such is difficult to obtain, therefore to make the equation (2.7) integrable, we use some judicious conditions on A or B or relation between A and B and then give resulting solution for e^A, e^B, r and p as functions of r which can be obtained by combining the new equation with (2.7), (2.3) and (2.4). Below we discuss the Schwarzschild interior solution obtained by Tolman in higher dimensions.

4. EINSTEIN UNIVERSE IN HIGHER DIMENSIONS

Here we choose

(4.1) e^B = constant = k (say)

With this assumption equation (2.7) becomes immediately integrable because the second two terms vanish due to the constancy of e^B and we get

$$\frac{d}{dr}\left[\frac{n(e^{-A}-1)}{r^2}\right]=0$$

(4.2)

which on integration yields

$$\frac{n(e^{-A}-1)}{r^2}=\text{constant}=-\frac{1}{R^2}\,(\text{say})$$

which gives

$$e^{-\lambda}=1-\frac{r^2}{nR^2}$$

(4.3)

Use of equation (4.1) in equations (2.4) and (4.3), we find pressure p as

$$p=\frac{(n+1)}{16\pi R^2}$$

(4.4)

Similarly, from equations (2.3) and (4.3), we find density ρ as

$$\rho=\frac{(n+2)(n+1)}{16\pi nR^2}$$

(4.5)

The resulting solution has found popularity in static cosmology. It is the analogue of static Einstein universe in higher dimension with uniform pressure and density.

5. SCHWARZSCHILD INTERIOR SOLUTION

In this case we assume

$$e^{-A} = \left[1 - \frac{r^2}{nR^2}\right] \tag{5.1}$$

which on substitution in (2.7) simplifies it since first term vanishes and we set

$$\frac{d}{dr}\left[\frac{e^{-A}B'}{2r}\right] + e^{-A-B}\frac{d}{dr}\left[\frac{e^{B}B'}{2r}\right] = 0$$

which on integration finally provides the solution

$$e^{B} = \left[\alpha - nkR^2\left(1 - \frac{r^2}{nR^2}\right)^{1/2}\right]^2 \tag{5.2}$$

where α and k are constants Now using equation (5.1) and (5.2) in (2.3) and (2.4) the pressure and density are found to be

$$p = \frac{(n+1)}{16\pi R^2}\left[(n+2)kR^2\left(1 - \frac{r^2}{nR^2}\right)^{1/2} - \alpha\right]\left[\alpha - nkR^2\left(1 - \frac{r^2}{nR^2}\right)^{1/2}\right]^{-1} \tag{5.3}$$

$$\rho = \frac{(n+1)(n+2)}{16\pi nR^2} \tag{5.4}$$

The solution can be considered as higher dimensional analogue of well-known Schwarzschild interior solution for a fluid sphere of constant density. When k = 0, the solution provides higher dimensional analogue of Einstein universe as obtained,

When $\alpha = 0$,

we find

$$e^{-A} = \left(1 - \frac{r^2}{nR^2}\right) \text{and } e^{B} = \text{constant}\left(1 - \frac{r^2}{nR^2}\right) \tag{5.5}$$

In this case we take

$$e^{B}\frac{B'}{2r} = \text{constant} \tag{5.6}$$

With this, the third term of equation (2.7) vanishes and we get

$$\frac{d}{dr}\left(\frac{e^{-A} - 1}{r^2}\right) + \frac{d}{dr}\left[\frac{e^{-A}B'}{2r}\right] = 0 \tag{5.7}$$

$$\Rightarrow \quad n\frac{(e^{-A} - 1)}{r^2} + \frac{e^{-A}B'}{2r} = \text{(constant)}$$

Integrating (5.6) we get

$$e^{B}=\mu^{2}\left[1+\frac{r^{2}}{\lambda^{2}}\right]$$
(5.8) (λ, μ are constants)

Which on combination with (5.7) finally gives
(5.9)

On substituting e^{-A}, A' and B' from (5.8) and (5.9) we can easily find p and ρ from equations (2.3) and (2.4) as (5.10)

$$\frac{16\pi}{(n+1)}\rho=\frac{1}{\lambda^{2}}\left[\frac{1+\frac{(n+2)\lambda^{2}}{nR^{2}}+\frac{(n+2)r^{2}}{nR^{2}}}{\left[1+\frac{(1+n)r^{2}}{n\lambda^{2}}\right]}\right]+\frac{2}{\lambda^{2}}\left[\frac{1-\frac{r^{2}}{nR^{2}}}{\left[1+\frac{(1+n)r^{2}}{n\lambda^{2}}\right]^{2}}\right]$$
(5.11)

The line element describing this solution can be written using e^{A} and e^{B} in equation (2.1)

At the centre of sphere, the pressure (p_C) and density (ρ_C) can be found by putting r = 0 in equations (5.10) and (5.11) and they are

$$p_{C}=\left(\frac{1}{A^{2}}-\frac{1}{R^{2}}\right)\left(\frac{(n+1)}{16\pi}\right)$$
(5.12)

and

$$\rho_{C}=\left(1+\frac{2}{n}\right)\left(\frac{1}{A^{2}}+\frac{1}{R^{2}}\right)\left(\frac{(n+1)}{16\pi}\right)$$
(5.13)

The equations (5.10) – (5.13) can be combined in the convent simple form given by

$$\rho=\rho_{C}-(n+4)(p_{C}-p)-\frac{4(n+1)(p_{C}-p)^{2}}{n(\rho_{C}+p_{C})}$$
(5.14)

Which is known as equation of state connecting the density and pressure of the fluid inside the sphere.

At the boundary of the sphere the pressure drops to zero and the boundary density ρ_b has the value for equation (5.14)

$$\rho_{b}=\rho_{C}-(n+4)p_{C}-\frac{4(n+1)p_{c}^{2}}{n(\rho_{C}+p_{C})}$$
(5.15)

From equation (5.10) we find that at the boundary $r = r_b$ of sphere (where p = 0), we have
(5.16)

The solution may be a useful one in studying properties of spheres of compressible fluid in higher dimensions since the equation of state (5.14) is relatively simple.

These solutions can be matched at the boundary $r = r_b$ with the exterior solution (Myers and Perry [20])
(5.17)

where w related to total mass of the fluid inside a sphere of radius r_b given by
(5.18)
were
(5.19)

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